Int. J. Therm. Sci. (1999) 38, 664–673

Radiative effects in natural convection flows

Mohamed Ahmed Mansour[a], Rama Subba Reddy Gorla[b*]

[a] *Mathematics Department, Faculty of Science, Assiut University, Assiut, Egypt*
[b] *Mechanical Engineering Department, Cleveland State University, Cleveland, Ohio 44115, USA*

(Received 17 August 1998, accepted 12 January 1999)

Abstract — A regular perturbation analysis is presented for the radiative effects on the following laminar natural convection flows with temperature dependent viscosity: a freely-rising plane plume, the flow above a horizontal line source on an adiabatic surface (a plane wall plume) and the flow adjacent to a vertical uniform flux surface. While these flows have well-known power-law similarity solutions when the fluid viscosity is taken to be constant, they are non-similar when the viscosity is considered to be a function of the temperature. A single similar flow, that adjacent to a vertical isothermal surface is also analyzed for comparison in order to estimate the accuracy. Numerical results with various parameters are tabulated. © Elsevier, Paris.

natural convection / radiative effects / heat transfer

Résumé — **Influence du rayonnement dans des écoulements de convection naturelle.** On présente l'analyse d'une perturbation régulière pour étudier les effets radiatifs sur divers écoulements laminaires en convection naturelle avec une viscosité dépendante de la température : un panache plan non confiné, un écoulement au-dessus d'une source linéaire horizontale sur une surface adiabatique (panache de paroi plane) et un écoulement adjacent à une surface verticale à flux uniforme. De tels écoulements ont des solutions bien connues, de type semblable, avec lois de puissance, lorsque la viscosité du fluide est constante, mais des profils non similaires lorsque la viscosité varie avec la température. Ainsi, un écoulement simple, de type similaire, adjacent à une surface verticale isotherme est aussi analysé pour valider le modèle. Les résultats numériques obtenus avec diverses valeurs des paramètres sont tabulés. © Elsevier, Paris.

convection naturelle / effets radiatifs / transferts de chaleur

Nomenclature

b, c, d	defined in equation (9)	
C_P	specific heat of the fluid	$\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$
C_T	temperature difference parameter	K
f	dimensionless stream function	
Gr_x	local Grashof number	
Gr'_x	actual local Grashof number	
g	acceleration due to gravity	$\mathrm{m \cdot s^{-2}}$
k	thermal conductivity	$\mathrm{W \cdot m^{-1} \cdot K^{-1}}$
K	absorption coefficient	$\mathrm{m^{-1}}$
M	momentum flux in the x direction	$\mathrm{kg \cdot s^{-2}}$
$\dot{m}$	mass flow rate per unit width of surface	$\mathrm{kg \cdot s^{-1} \cdot m^{-1}}$
Nu_x	Nusselt number	
N'	heat transfer parameter	
Pr	Prandtl number	
Q	total heat convected downstream	$\mathrm{W \cdot m^{-2}}$
q^*	surface heat flux	$\mathrm{W \cdot m^{-2}}$
q^e	local radiative heat flux	$\mathrm{W \cdot m^{-2}}$
R	conduction–radiation parameter	
T	temperature	K
T_f	film temperature	K
u, v	velocity components in x and y directions	$\mathrm{m \cdot s^{-1}}$

Greek symbols

α	thermal diffusivity	$\mathrm{m^2 \cdot s^{-1}}$
β	coefficient of thermal expansion	$\mathrm{K^{-1}}$
η	dimensionless distance	
θ	dimensionless temperature	m
μ	dynamic viscosity	$\mathrm{kg \cdot m^{-1} \cdot s^{-1}}$
ρ	density	$\mathrm{kg \cdot m^{-3}}$
γ	perturbation parameter	
ψ	stream function	$\mathrm{m^2 \cdot s^{-1}}$
τ	shear stress	$\mathrm{kg \cdot m^{-1} \cdot s^{-2}}$

* Correspondence and reprints.
gorla@csaxp.csuohio.edu

Subscripts

f condition at film temperature
0 condition at the wall
∞ condition far away from the surface

Superscripts

$'$ differentiation with respect to η

1. INTRODUCTION

Due to the importance of the influence of the variable fluid properties in many engineering applications, a lot of analytical and experimental work has been directed towards determining the effects of variable fluid-properties in natural convection flows. Much of this work was reviewed in the recent papers by Kakac et al. [1], Herwig [2] and Kakac [3].

The earliest theoretical treatment of variable property effects on natural convection is the analysis of Hara [4] for air. Sparrow [5, 6] considered natural convection with variable properties. He indicated that the film temperature is adequate for most applications and suggested a more accurate reference temperature for more extreme conditions. Variable property effects in water and carbon dioxide at supercritical pressures were analyzed by Nishikawa and Ito [7] in the case of natural convection adjacent to a vertical isothermal surface. Barrow and Rao [8] examined the effect of variable coefficient of thermal expansion, β, on natural convection in water, but ignored the temperature dependence of absolute viscosity μ, which is known to be important. Brown [9] used an integral method with variable β and the density, ρ, but overlooked the important variation of μ in his study of natural convection.

Carey and Mollendorf [10] presented a boundary layer similarity analysis for the laminar natural convection from a vertical isothermal surface. Their analysis was applicable to liquids wherein the viscosity variations are large compared to other fluid properties. Thermal radiation effect on natural convection in laminar boundary layer flow with constant properties over an isothermal flat plate was investigated by Ali et al. [11]. The radiative mode of heat transfer becomes important when the temperature difference between the plate surface and ambient is large. The motivation for the current study came from a lack of understanding in the literature, of the coupled effects of variable viscosity and thermal radiation on natural convection heat transfer.

The present study was undertaken in order to investigate the effects of thermal radiation and variable viscosity on some natural convection flows. Results have been obtained for four representative kinds of surface temperature variation, namely, an isothermal surface, a uniform heat flux surface, a plane plume and the flow generated from a horizontal line energy source on a vertical adiabatic surface.

2. ANALYSIS

Let us consider a steady, two-dimensional, vertical natural convection flow and incorporate the usual Boussinesq and boundary layer assumptions. The absolute viscosity, μ, is taken to be a temperature-dependent variable in the force-momentum balance. The fluid is assumed to be a gray, emitting and absorbing, but non-scattering medium. The radiative heat flux in the x direction is considered negligible in comparison with that in the y-direction. This results in the following governing equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \frac{1}{\rho}\,\frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right) + g\,\beta\,(T - T_\infty) \tag{2}$$

$$u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial y^2} - \frac{1}{k}\,\frac{\partial q^r}{\partial y}\right) \tag{3}$$

where u and v are the vertical and horizontal velocity components respectively and T is the temperature. Except for μ, the fluid properties in (2) and (3) are as viewed as constant to be evaluated at some reference temperature.

The radiative heat flux term is simplified by using the Rosseland approximation [12] as:

$$q^{\mathrm{r}} = -\frac{4\,\sigma}{3\,K}\,\frac{\partial T^4}{\partial y} \tag{4}$$

where σ and K are the Stefan–Boltzman constant and mean absorption coefficient, respectively.

Proceeding with the analysis, we introduce the following transformations:

$$\begin{aligned}
&\eta = y\,b(x), \psi = \frac{\mu_{\mathrm{f}}}{\rho}\,c(x)\,f(\eta, x)\\
&\theta(\eta, x) = \frac{T - T_\infty}{(T_0 - T_\infty)}, \ (T_0 - T_\infty)_0 = d(x) = N\,x^n\\
&c(x) = 4\,x\,b(x) = 4\left[\frac{g\,\beta\,\rho^2\,x^3\,(T_0 - T_\infty)_0}{4\,\mu_{\mathrm{f}}}\right]^{1/4}\\
&= 4\left[\frac{Gr_x}{4}\right]^{1/4}, \ R = \frac{4\,\sigma\,(T_0 - T_\infty)}{k\,K}, \ C_T = \frac{T_\infty}{(T_0 - T_\infty)_0}
\end{aligned} \tag{5}$$

where R is the conduction-radiation parameter, C_{T} is the temperature difference parameter, and $(T_0 - T_\infty)$ is the downstream temperature difference (along the x-axis). The absolute viscosity, μ, is assumed to vary with temperature according to a general functional form

$\mu = \mu_f\, S(\theta)$, where μ_f is the absolute viscosity at the film temperature T_f, where $\theta = 0.5$. Therefore, $S(1/2) = 1$. This form is chosen to allow definition of the stream function based on the absolute viscosity at the film temperature. For all liquids, all transport properties vary with temperature. However, for many liquids, such as petroleum oils, glycerin, glycols, silicone fluids and some molten salts, the percent variation of absolute viscosity with temperature is much greater than that of the other properties. Under these conditions, an analysis incorporating the above assumptions describes the momentum and thermal transport within the flow field much more accurately than the usual assumption of constant properties, evaluated at some reference temperature. A wide variety of functional forms of $S(\theta)$ satisfy this requirement, for example, algebraic expressions, power series, exponential forms etc. A simple but accurate form of S is considered here, namely:

$$S = \left[1 + \frac{1}{\mu_f}\,\left(\frac{\mathrm{d}\mu}{\mathrm{d}T}\right)\,(T - T_f)\right] \tag{6}$$

This simple form amounts to a linear variation of absolute viscosity with temperature, with the slope $\mathrm{d}\mu/\mathrm{d}T$ evaluated at the film temperature.

The assumed linear variation of viscosity with temperature gives rise to a new parameter γ_f, defined as:

$$\gamma_f = \left(\frac{1}{\mu}\,\frac{\mathrm{d}\mu}{\mathrm{d}T}\right)_f (T_0 - T_\infty) \tag{7}$$

where μ_f is the value of μ at the film temperature. Substituting equation (7) into equation (6), we have the following:

$$S = [1 + \gamma_f\,(\theta - 0.5)] \tag{8}$$

For a linear variation of viscosity with temperature, if μ_0 and μ_∞ are the values of viscosity evaluated at the surface and ambient temperatures, respectively, then we have:

$$\gamma_f = 2\,(\mu_0 - \mu_\infty)/(\mu_0 + \mu_\infty) \tag{9}$$

It can be seen from equation (9) that the maximum value of 2 for γ_f occurs when (μ_0/μ_∞) approaches infinity. Likewise, the minimum value of -2 for γ_f occurs when (μ_0/μ_∞) approaches zero. Piau [13] suggested that for higher Prandtl number liquids, the variation of β with temperature is negligible. He performed calculations for the asymptotic case, $Pr \to \infty$ for a single data point involving $\gamma_f = -0.3614$.

Using equation (8), we may write:

$$\mu = \mu_f\left[1 + \gamma_f\left(\theta - \frac{1}{2}\right)\right] \tag{10}$$

Expansions for the dimensionless stream function and temperature are postulated as:

$$f(\eta,\gamma_f) = f(\eta, x) = f_0(\eta) + \gamma_f\, f_1(\eta) + \ldots \tag{11}$$

$$\theta(\eta,\gamma_f) = \theta(\eta, x) = \theta_0(\eta) + \gamma_f\,\theta_1(\eta) + \ldots \tag{12}$$

Here, we consider only first order terms and therefore the expansion for μ, f and θ are truncated after terms of order γ_f.

Substituting (10), (11) and (12) into (2) and (3) and using the transformations in (5), the equations for f_0, θ_0, f_1 and are then obtained as follows:

$$f_0''' - (2n+2)\,f_0'^2 + (n+3)\,f_0''\,f_0 + \theta_0 = 0 \tag{13}$$

$$\left[1 + \frac{4}{3}\,R\,(C_T + \theta_0)^3\right]\,\theta_0'' + 4\,R\,(C_T + \theta_0)^2\,\theta_0'^2 + Pr\,\left[(n+3)\,\theta_0'\,f_0 - 4\,n\,f_0'\,\theta_0\right] = 0 \tag{14}$$

$$f_1''' - (8n+4)\,f_0'\,f_1' + (5n+3)\,f_0''\,f_1 + (n+3)\,f_0\,f_1'' + \theta_1 + f_0''\,(\theta_0 - 1/2) + \theta_0'\,f_0'' = 0 \tag{15}$$

$$\begin{aligned}&\left[1 + \frac{4R}{3}\,(C_T + \theta_0)^3\right]\,\theta_1'' + 4\,R\,(C_T + \theta_0)^2\,\theta_1\,\theta_0'' \\ &+ 8\,R\,(C_T + \theta_0)^2\,\theta_0'\,\theta_1' + 8\,R\,(C_T + \theta_0)\,\theta_1\,\theta_0'^2 \\ &+ Pr\,\big[(n+3)\,f_0\,\theta_1' - 8\,n\,f_0'\,\theta_1 - 4\,n\,f_1'\,\theta_0 \\ &+ (5n+3)\,f_1\,\theta_0'\big] = 0\end{aligned} \tag{16}$$

In the above equations, a prime indicates differentiation with respect to η only. The relevant boundary conditions for the four flows to be analyzed here are as follows:

a) isothermal surface with horizontal leading edge $n = 0$

$$\theta(\infty,x) = f'(0,x) = f(0,x) = f'(\infty,x) = 1 - \theta(0,x) = 0$$
$$1 - \theta_0(0) = \theta_0(\infty) = f_0'(0) = f_0(0) = f_0'(\infty) = 0$$
$$\theta_1(0) = \theta_1(\infty) = f_0'(0) = f_1(0) = f_1'(\infty) = 0 \tag{17}$$

b) uniform flux surface with a horizontal leading edge, $n = 1/5$

$$\theta(\infty,x) = f'(0,x) = f(0,x) = f'(\infty,x) = 0$$
$$1 - \theta_0(0) = \theta_0'(0) = f_0'(0) = f_0(0) = f_0'(\infty) = 0$$
$$\theta_1'(0) = \theta_1(\infty) = f_1'(0) = f_1(0) = f_1'(\infty) = 0 \tag{18}$$

c) an adiabatic surface with a concentrated heat source along the horizontal leading edge, $n = -3/5$.

$$\theta'(0,x) = f'(0,x) = f(0,x) = f'(\infty,x) = 0$$
$$1 - \theta_0(0) = \theta_0'(0) = f_0'(0) = f_0(0) = f_0'(\infty) = 0$$
$$\theta_1'(0) = \theta_1(\infty) = f_1'(0) = f_1(0) = f_1'(\infty) = 0 \tag{19}$$

d) a plane plume rising from a horizontal thermal source, $n = -3/5$

$$\theta'(0,x) = f(0,x) = f''(0,x) = f'(\infty,x) = 0$$
$$1 - \theta_0(0) = \theta_0'(0) = f_0(0) = f_0''(0) = f_0'(\infty) = 0$$
$$\theta_1'(0) = \theta_1(\infty) = f_1'(0) = f_1(0) = f_1'(\infty) = 0 \tag{20}$$

For the isothermal condition, $n = 0$, and since $\theta_1(0) = 0$, the temperature at $y = 0$ is not altered by varying γ_f. Consequently, the film temperature, $T_f = (T_0 + T_\infty)/2$, and $(T_0 - T_\infty)$ are not altered by varying γ_f. Therefore, for the isothermal condition, γ_f is equal to $\left(\frac{1}{\mu}\right)_f \left(\frac{d\mu}{dT}\right)_f (T_0 - T_\infty)$ as defined by Carey and Mollendorf [4]. The values of n shown above for the other three flow conditions are determined by calculating the value of $Q(x)$, the total heat convected in the flow at any downstream location x:

$$Q(x) = \int_0^\infty \rho C_p (T - T_\infty) u \, dy$$
$$= \mu_f C_p c d \int_0^\infty \theta f \, d\eta \propto x^{(3+5n)/4} \qquad (21)$$

This must increase linearly with x for the uniform heat flux surface condition, (b), and independently of x for the adiabatic flows, (c) and (d). Therefore,

$$n_a = 0, \; n_b = 1/5, \; n_c = n_d = -3/5 \qquad (22)$$

Including the first order terms in f and θ for $\gamma_f \neq 0$, $Q(x)$ is:

$$Q(x) = \mu_f C_p c d \left[\int_0^\infty \theta_0 f_0' \, d\eta + \gamma_f \int_0^\infty \left(\theta_0 f_1' + \theta f_0'\right) d\eta \right] \qquad (23)$$

For the uniform flux condition, (b), and the adiabatic flows, (c) and (d), integration of the first order energy equation (16) shows that the second integral in (23) is zero. This is required to ensure that additional x dependence is not added to $Q(x)$ through γ_f. The mass flow per unit width of surface, $\dot{m}$, becomes

$$\dot{m} = \int_0^\infty \rho u \, dy$$
$$= \mu_f c \int_0^\infty f' \, d\eta = \mu_f c \left[f_0(\infty) + \gamma_f f_1(\infty)\right] \qquad (24)$$

and the momentum flux in the x direction is given by:

$$M(x) = \int_0^\infty \rho u^2 \, dy = \frac{\mu_f^2 c^2 b}{\rho} \int_0^\infty f'^2 \, dy$$
$$= \frac{\mu_f^2 c^2 b}{\rho} \left[I_{M0} + \gamma_f I_{M1}\right] \qquad (25)$$

where

$$I_{M0} = \int_0^\infty f_0'^2 \, d\eta \quad \text{and} \quad I_{M1} = \int_0^\infty 2 f_0' f_1' \, d\eta$$

The stream function, as defined here, is based on the film viscosity. For the flows adjacent to a vertical surface, the shear stress at the surface, $\tau_0(x)$, is therefore a function of γ_f directly, as well as through $f''(0)$:

$$\tau_0(x) = \mu_f^2 (4/x^2) \left(Gr_x'/4\right)^{3/4} \tau^*/\rho \qquad (26)$$

where $\tau^* = (1 + \gamma_f \theta(0)/2) f''(0)/[\theta(0)]^{3/4}$.

Substituting the expansions for θ, f'' and keeping only first order terms in γ_f yields

$$\tau^* = \left[\left(1 + \gamma_f/2\right) f_0''(0) + \gamma_f f_1''(0)\right] \Big/ \left[\theta_0(0) + \gamma_f \theta_1(0)\right]^{3/4} \qquad (27)$$

The surface heat flux, $q''(x)$ and the local Nusselt number Nu_x are determined as follows:

$$q'' = -k \left(\frac{\partial T}{\partial y}\right)_{y=0} - \frac{4\sigma}{3K} \left(\frac{\partial T^4}{\partial y}\right)_{y=0} \qquad (28)$$

$$Nu_x = \frac{q''}{(T_0 - T_\infty)} \frac{x}{k}$$
$$= -\frac{\theta'(0)}{[\theta(0)]^{5/4}} \frac{(Gr_x')}{\sqrt{2}} \left[1 + \frac{4R}{3}(C_T + \theta)^3\right] \qquad (29)$$

where $\theta(0) = \theta_0(0) + \gamma_f \theta_1(0)$ and $Gr_x' = Gr_x \theta(0)$.

Defining the heat transfer parameter N' as:

$$N' = \frac{Nu_x \sqrt{2}}{(Gr_x')^{1/4}}$$

we have:

$$N' = \frac{[-\theta'(0)]}{[\theta(0)]^{5/4}} \left[1 + \frac{4R}{3}(C_T + \theta)^3\right] \qquad (30)$$

3. RESULTS AND DISCUSSION

The system of equations (13–16) with the boundary conditions (a–d) was solved numerically by the fourth order Runge-Kutta integration scheme. Calculations were carried out for the values of Prandtl number of 5, 10 and 50. The conduction-radiation parameter R ranged from 0 to 1 with $C_T = 0.1$.

Figures 1 and *2* illustrate the velocity and temperature profiles, for the isothermal surface condition. We observe that the velocity f' and temperature θ within the boundary layer increase with increasing values of R. As the temperature difference between the surface and ambient $(T_0 - T_\infty)$ increases, the buoyancy force increases, which in turn will augment the streamwise velocity. We also note that as $(T_0 - T_\infty)$ increases, the conduction–radiation parameter, R, increases and therefore augments the temperature distribution within the boundary layer.

Figure 3 shows the values of the heat transfer parameter, N', predicted by the perturbation analysis for the isothermal and uniform heat flux surfaces. For both the isothermal and uniform heat flux surface conditions, $\gamma_f < 0$ increases the surface heat transfer while $\gamma_f > 0$ reduces it. Also, the heat transfer parameter N' increases as R decreases.

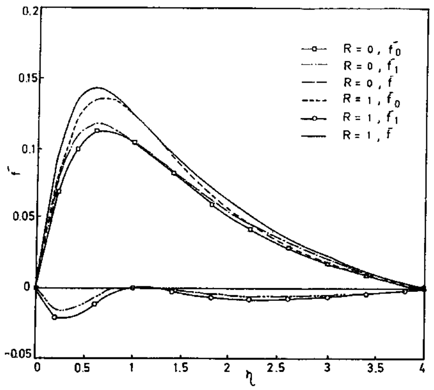

Figure 1. Velocity profile for the isothermal surface condition with $Pr = 10$ and $\gamma_f = -0.8$.

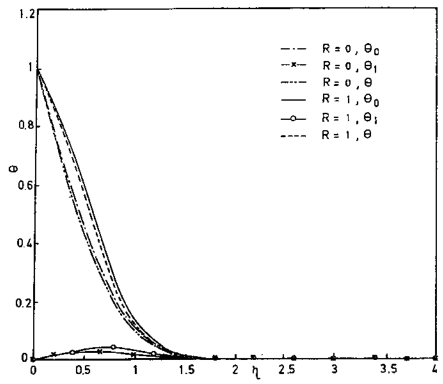

Figure 2. Temperature profile for the isothermal surface condition with $Pr = 10$ and $\gamma_f = -0.8$.

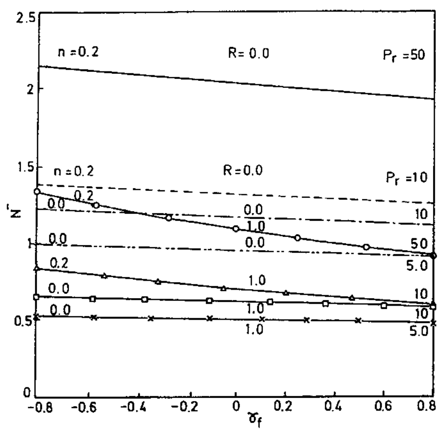

Figure 3. The effect of gf on heat transfer for the isothermal and uniform heat flux surfaces.

Figures 4–7 show the effects of R and γ_f on the velocity and temperature profiles for the flow above a horizontal line thermal source and those on a vertical adiabatic surface. We notice that the velocity and temperature increase with increasing values of R. The effect of γ_f is similar to that of R. For $\gamma_f < 0$, increasing

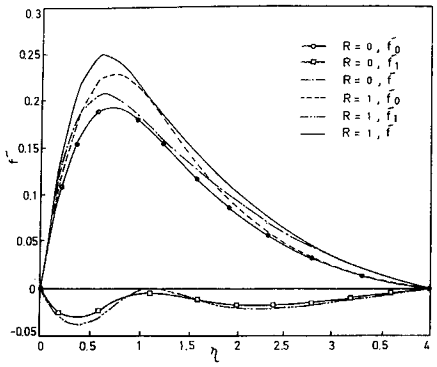

Figure 4. Velocity profile for an adiabatic surface condition with $Pr = 10$ and $\gamma_f = -0.8$.

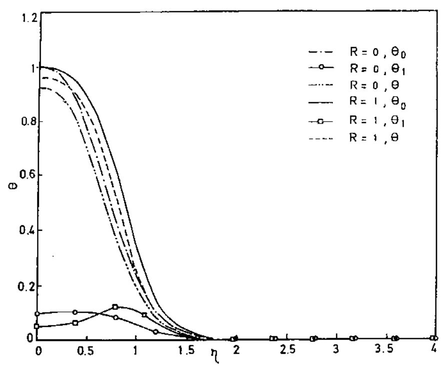

Figure 5. Temperature profile for an adiabatic surface condition with $Pr = 10$ and $\gamma_f = -0.8$.

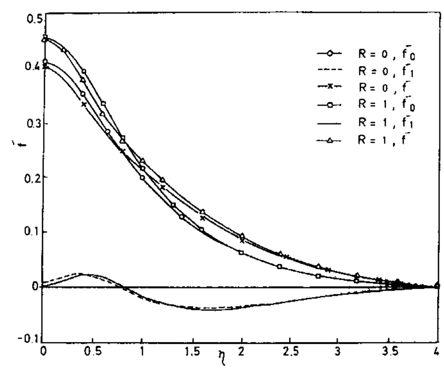

Figure 6. Velocity Profile for the plane plume surface condition with $Pr = 10$ and $\gamma_f = -0.8$.

R produces increasing $\theta(0, \gamma_f)$, while for $\gamma_f > 0$, the effect is opposite. We note that for many liquids, β is greater than zero and $\gamma_f = \left(\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}T}\right)_f (T_0 - T_\infty)$ is less than zero. The most common case for $\gamma_f < 0$ is given by $T_0 > T_\infty$ and upward flow with $\mu_0 < \mu_\infty$. The most common case for $\gamma_f > 0$ is given by $T_0 < T_\infty$ and upward flow with $\mu_0 > \mu_\infty$.

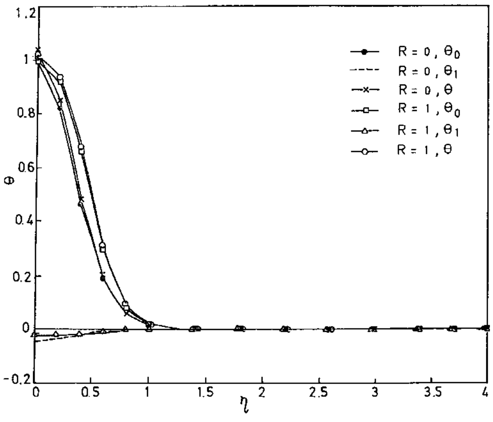

Figure 7. Temperature profile for the plane plume surface condition with $Pr = 10$ and $\gamma_f = -0.8$.

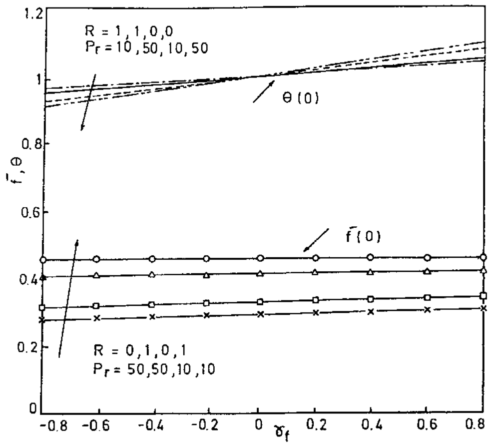

Figure 8. The temperature (adiabatic surface) and the velocity (plane plume) at the wall.

Figure 8 shows the effect of γ_f, Pr and R on the centerline velocity and temperature for the plane plume. Increasing γ_f actually produces a slight decrease in $f'(0, \gamma_f)$ and an increase in the centerline temperature for the adiabatic flow.

The numerical results of the perturbation analysis for the four flow configurations corresponding to the indicated values of Pr, γ_f and R are summarized in *tables I–IV*. For the isothermal condition, *table V* provides a detailed comparison of the perturbation results with those for the corresponding similarity solution.

4. CONCLUDING REMARKS

The present results show the effect of a linear temperature dependent viscosity on laminar natural convection in liquids when other property variations can be neglected. The temperature difference between the surface and the ambient is assumed to be large enough to include radiative effects. The assumption that other property variations are small when compared to the viscosity variation is realistic for many liquids. The assumption of a linear dependence of viscosity

TABLE I
The numerical results of the perturbation analysis. a) isothermal case $n = 0$.

R	Pr	γ_f	$f_0''(0)$	$f_1''(0)$	$\theta_0'(0)$	$\theta_1'(0)$	$f''(0)$	$\theta'(0)$
0.0	5.0	−0.8	0.48047	−0.18265	−0.95103	0.05581	0.62659	−0.99568
		0.0	–	–	–	–	0.48047	−0.95103
		0.8	–	–	–	–	0.33435	−0.90638
5.0	5.0	−0.8	0.52044	−0.18940	−0.62612	0.03914	0.67196	−0.65743
–	–	0.0	–	–	–	–	0.52044	−0.62612
–	–	0.8	–	–	–	–	0.36892	−0.59481
1.0	5.0	−0.8	0.54936	−0.19482	−0.50168	0.03172	0.70521	−0.52705
–	–	0.0	–	–	–	–	0.54936	−0.50168
–	–	0.8	–	–	–	–	0.39350	−0.47630
0.0	10	−0.8	0.41767	−0.16238	−1.16521	0.07222	0.54757	−1.22299
–	–	0.0	–	–	–	–	0.41767	−1.16521
–	–	0.8	–	–	–	–	0.28777	−1.10743
0.5	10	−0.8	0.45390	−0.16930	−0.76870	0.05133	0.58935	−0.80976
–	–	0.0	–	–	–	–	0.45390	−0.76870
–	–	0.8	–	–	–	–	0.31846	−0.76870
1.0	10	−0.8	0.48057	−0.17483	−0.61727	0.04201	0.62043	−0.65088
–	–	0.0	–	–	–	–	0.48057	−0.61727
1.0	10	0.8	0.48057	−0.17483	−0.61727	0.04201	0.34071	−0.58366

TABLE II
Uniform heat flux $n = 0.2$.

R	Pr	γ_f	$f_0''(0)$	$f_1''(0)$	$\theta_0'(0)$	$\theta_1(0)$	$f''(0)$	$\theta(0)$
0.0	5.0	−0.8	0.45329	−0.15830	−1.07333	0.04749	0.57993	0.96201
–	–	0.0	–	–	–	–	0.45329	1.00000
–	–	0.8	–	–	–	–	0.32664	1.03799
5.0	5.0	−0.8	0.49181	−0.13134	−0.70803	0.11434	0.59688	0.90853
–	–	0.0	–	–	–	–	0.49181	1.00000
–	–	0.8	–	–	–	–	0.38674	1.09147
1.0	5.0	−0.8	0.51978	−0.10758	−56830	0.16004	0.60584	0.87197
–	–	0.0	–	–	–	–	0.51978	1.00000
–	–	0.8	–	–	–	–	0.433710	1.12803
0.0	10	−0.8	0.39370	−0.14012	−1.31228	0.05008	0.50579	0.95993
–	–	0.0	–	–	–	–	0.39370	1.00000
–	–	0.8	–	–	–	–	0.28160	1.04007
0.5	10	−0.8	0.42852	−0.11544	−0.86726	0.12108	0.52086	0.90314
–	–	0.0	–	–	–	–	0.42852	1.00000
–	–	0.8	–	–	–	–	0.33617	1.09686
1.0	10	−0.8	0.45421	−0.09363	−0.69751	0.16840	0.52912	0.86528
–	–	0.0	–	–	–	–	0.45421	1.00000
1.0	10	0.8	0.45421	−0.09363	−0.69751	0.16840	0.37931	1.13472

on temperature gives rise to the parameter γ_f whose value dictates the magnitude of the viscosity variation. The results indicate that the temperature dependent viscosity has a significant effect on the temperature and velocity fields as well as the heat transfer rate and drag. The strong effect of temperature dependent viscosity on the flow field would suggest the possibility of significant effects on the stability and transition of such flows.

Acknowledgments

The authors are grateful to the reviewers for their useful comments.

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TABLE III
Line source plume $n = -3/5$.

R	Pr	γ_f	$f_0''(0)$	$f_1''(0)$	$\theta_1(0)$	$f'(0)$	$\theta(0)$	I_Q	I_{M0}	I_{M1}
0	4	−0.8	0.4739	−0.0036	−0.0147	0.4768	1.0118	0.2256	0.1703	−0.00892
–	–	0.0	–	–	–	0.4739	1.0000	0.2313	0.1745	−0.00914
–	–	0.8	–	–	–	0.4712	0.9882	0.2314	0.1746	−0.00914
0.5	5	−0.8	0.5008	−0.0088	−0.0114	5 0.5078	1.0091	0.2699	0.1944	−0.00806
–	–	0.0	–	–	–	0.5008	1.0000	0.2709	0.1949	−0.00803
–	–	0.8	–	–	–	0.4938	0.9909	0.2709	0.1949	−0.00803
1	5	−0.8	0.5215	−0.0131	−0.0102	0.5319	1.0082	0.3041	0.2120	−0.00698
–	–	0.0	–	–	–	0.5215	1.0000	0.3049	0.2124	−0.00695
–	–	0.8	–	–	–	0.5110	0.9918	0.3049	0.2124	−0.00695
0	10	−0.8	0.4139	0.0028	−0.0244	0.4117	1.0195	0.1614	0.1326	−0.01137
–	–	0.0	–	–	–	0.4139	1.0000	0.1579	0.1306	−0.01148
–	–	0.8	–	–	–	0.4162	0.9805	0.1578	0.1306	−0.01148
0.5	10	−0.8	0.4387	−0.0007	−0.0168	0.4393	1.0135	0.1847	0.1451	−0.01117
–	–	0.0	–	–	–	0.4387	1.0000	0.1853	0.1454	−0.01116
–	–	0.8	–	–	–	0.4382	0.9865	0.1854	0.1454	−0.01116
1	10	−0.8	0.4580	−0.0038	−0.0140	0.4610	1.0112	0.2087	0.1579	−0.01065
–	–	0.0	–	–	–	0.4580	1.0000	0.2093	0.1582	−0.01064
–	–	0.8	–	–	–	0.4550	0.9888	0.2093	0.1582	−0.01064
0	50	−0.8	0.2935	0.0105	−0.0461	0.2851	1.0369	0.0654	0.0705	−0.01111
–	–	0.0	–	–	–	0.2935	1.0000	0.0619	0.0683	−0.01112
–	–	0.8	–	–	–	0.3019	0.9631	0.0618	0.0683	−0.01112
0.5	50	−0.8	0.3126	0.0096	−0.0302	0.3049	1.0241	0.0726	0.0757	−0.01186
–	–	0.0	–	–	–	0.3126	1.0000	0.0729	0.0758	−0.01187
–	–	0.8	–	–	–	0.3203	0.9759	0.0729	0.0758	−0.01188
1	50	−0.8	0.3278	0.0087	−0.0242	0.3208	1.0194	0.0825	0.0821	−0.01233
–	–	0.0	–	–	–	0.3278	1.0000	0.0827	0.0828	−0.01234
1	50	0.8	0.3278	0.0087	−0.0242	0.3348	0.9806	0.0827	0.0823	−0.01234

TABLE IV
Line source on adiabatic surface $n = -3/5$.

R	Pr	γ_f	$f_0''(0)$	$f_1''(0)$	$\theta_1(0)$	$f''(0)$	$\theta(0)$	I_Q	I_{M0}	I_{M1}
0	5	−0.8	0.6923	−0.1901	0.1127	0.8444	0.9098	0.1689	0.0781	−0.02344
–	–	0.0	–	–	–	0.6923	1.0000	0.1731	0.0800	−0.02403
–	–	0.0	–	–	–	0.6923	1.0000	0.1731	0.0800	−0.02403
–	–	0.8	–	–	–	0.3403	1.0902	0.1732	0.0801	−0.02405
0.5	5	−0.8	0.7306	−0.2092	0.0687	0.8980	0.9450	0.2074	0.0930	−0.02478
–	–	0.0	–	–	–	0.7306	1.0000	0.2082	0.0934	−0.02480
–	–	0.8	–	–	–	0.5632	1.0550	0.2083	0.0934	−0.02480
1	5	−0.8	0.7589	−0.2207	0.0527	0.9355	0.9679	0.2386	0.1053	−0.02528
–	–	0.0	–	–	–	0.7589	1.0000	0.2394	0.1056	−0.02530
–	–	0.8	–	–	–	0.5823	1/0421	0.2394	0.1056	−0.02530
0	10	−0.8	0.6106	−0.1688	0.1247	0.7456	0.9003	0.1126	0.0516	−0.01886
–	–	0.0	–	–	–	0.6106	1.0000	0.1094	0.0502	−0.01869
–	–	0.8	–	–	–	0.4755	1.0997	0.1094	0.0502	−0.01869
0.5	10	−0.8	0.6469	−0.1863	0.0776	0.7959	0.9379	0.1315	0.0584	−0.02019
–	–	0.0	–	–	–	0.6469	1.0000	0.1321	0.0586	−0.02023
–	–	0.8	–	–	–	0.4979	1.0621	0.1321	0.0586	−0.02023
1	10	−0.8	0.6743	−0.1967	0.0606	0.8317	0.9515	0.1520	0.0663	−0.02143
–	–	0.0	–	–	–	0.6743	1.0000	0.1525	0.0665	−0.02146
–	–	0.8	–	–	–	0.5169	1.0485	0.1525	0.0665	−0.02146
	50	−0.8	0.4393	−0.1245	0.1426	0.5389	0.8859	0.0386	0.0173	−0.00804
–	–	0.0	–	–	–	0.4393	1.0000	0.0355	0.0161	−0.00770
–	–	0.8	–	–	–	0.3397	1.1141	0.0354	0.0161	−0.00769
0.5	50	−0.8	0.4686	−0.1383	0.0911	0.5792	0.9271	0.0429	0.0189	−0.00897
–	–	0.0	–	–	–	0.4686	1.0000	0.0431	0.0189	−0.00901
–	–	0.8	–	–	–	0.3580	1.0729	0.0431	0.0189	−0.00901
1	50	−0.8	0.4914	−0.1467	0.729	0.6088	0.9417	0.0526	0.0227	−0.01047
–	–	0.0	–	–	–	0.4914	1.0000	0.0501	0.0216	−0.01020
1 –	50 –	0.8 0.0	0.4914 –	−0.1467 –	0.0729 –	0.6088 0.4914	0.9417 1.0000	0.0526 0.0501	0.0227 0.0216	−0.01047 −0.1020
1	50	0.8	0.4914	−0.1467	0.0729	0.3741	1.0583	0.0500	0.0216	−0.01019

TABLE V
Comparison of the perturbation analysis, p, with the similarity solution, s, for an isothermal vertical surface.

R	Pr	γ_f	$\frac{f_s''(0)}{f_p''(0)}$	% Error	$\frac{\theta_s'(0)}{\theta_p'(0)}$	% Error
0.0	5	−0.8	0.71246 0.62910	13.3	−1.01301 −1.00022	1.3
–	–	0.8	0.37234 0.33415	11.4	−0.91483 −0.90704	0.9
0.0	10	−0.8	0.62556 0.55028	13.7	−1.24599 −1.22890	14
–	–	0.8	0.32187 0.28760	11.9	−1.11884 −1.10840	0.9
0.3	5	−0.8	0.74191 0.65819	12.7	−0.76090 −0.75105	1.3
–	–	0.8	0.39518 0.35652	10.8	−0.68406 −0.67807	0.9
0.30	10	−0.8	0.65299 0.57325	13.3	−0.93766 −0.92315	1.6
–	–	0.8	0.34223 0.30746	11.3	−0.83723 −0.82903	1.0